The Complexity of Finding the Next Whisky Bar

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Outline

Suppose we have a map with whisky bars marked on it.

How difficult is it to find the next whisky bar

- . . . if you are anywhere on the map?
- . . . if you are already in a whisky bar?

How difficult is it to determine the minimal distance between any two whisky bars?

Constraint Satisfaction Problems

R . . . set of relations over some domain D

CSP(R)

Instance: formula $\varphi = r_1(\boldsymbol{x}_1) \wedge \cdots \wedge r_n(\boldsymbol{x}_n)$ where $r_i \in R$
Question: Is φ satisfiable?

The relations in R can be used to define further relations.

Relational clone generated by relations R

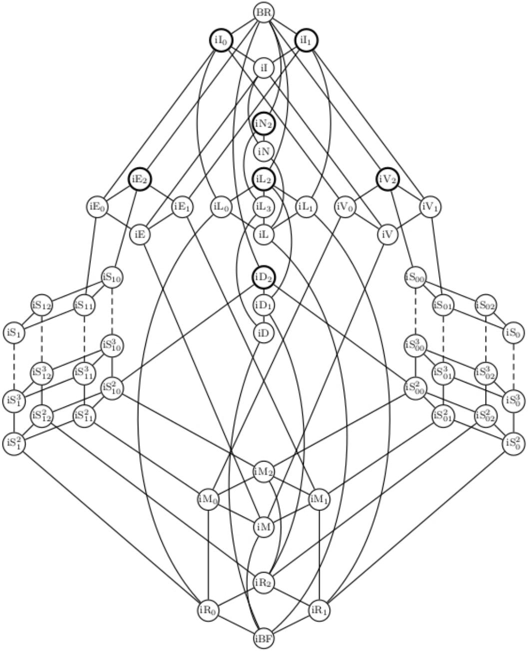
$\langle R \rangle$ is the smallest set containing R such that

- $=_D \in \langle R \rangle$
- $\langle R \rangle$ is closed under conjunction and existential quantification

If $p/2, q/2 \in R$, then $r(x) := \exists z\, (p(x, z) \wedge q(z, x)) \in \langle R \rangle$.

If $\langle R \rangle = \langle R' \rangle$, then CSP($R$) and CSP($R'$) are polynomial-time equivalent.

Post's lattice of Boolean relational clones



$\mathsf{BR} = \langle \{1\mathrm{in}3\} \rangle$

$\mathsf{iE}_2 = \langle \{\neg x \vee \neg y \vee z, \neg x, x\} \rangle$

$\mathsf{iV}_2 = \langle \{x \vee y \vee \neg z, \neg x, x\} \rangle$

$\mathsf{iBF} = \langle \{x \equiv y\} \rangle$

Polymorphisms and invariants

A function f/k is a polymorphism of a relation r/n (r is an invariant of f), if the following rule holds:

$$\frac{\begin{array}{c}(d_{11}, \dots, d_{1n}) \in r \\ \vdots \\ (d_{k1}, \dots, d_{kn}) \in r\end{array}}{(f(d_{11}, \dots, d_{k1}), \dots, f(d_{1n}, \dots, d_{kn})) \in r}$$

$\mathrm{Pol}\, r := \{\, f \mid f \text{ is a polymorphism of } r \,\}$ $\mathrm{Pol}\, R := \bigcap_{r \in R} \mathrm{Pol}\, r$

$\mathrm{Inv}\, f := \{\, r \mid r \text{ is an invariant of } f \,\}$ $\mathrm{Inv}\, F := \bigcap_{f \in F} \mathrm{Inv}\, f$

Minimum is a polymorphism of $\mathrm{iE}_2 = \langle \{\neg x \vee \neg y \vee z, \neg x, x\} \rangle$.

Projections are the only polymorphisms of $\mathrm{BR} = \langle \{1\mathrm{in}3\} \rangle$.

Every function is a polymorphism of $\mathrm{iBF} = \langle \{x \equiv y\} \rangle$.

Observation:

- Projections are polymorphisms of all relations.
- If f and f' are polymorphisms of a relation, then their composition is also a polymorphism.

Functional clone generated by functions F

$[F]$ is the smallest set containing F such that

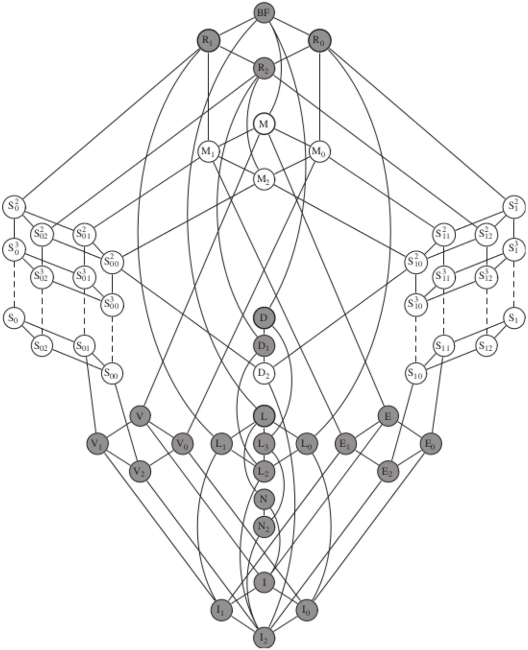
- all projections are in $[F]$
- $[F]$ is closed under composition

$\mathrm{L} = [\{x \oplus y, 1\}]$ is the set of Boolean linear functions.

$\mathrm{BF} = [\{\mathrm{nand}\}]$ is the set of all Boolean functions.

$\mathrm{Inv}\,\mathrm{Pol}\,R = \langle R \rangle \qquad \mathrm{Pol}\,\mathrm{Inv}\,F = [F]$

Post's lattice of Boolean functional clones



$\mathsf{BF} = [\{\mathsf{nand}\}]$

$\mathsf{L} = [\{x \oplus y, 1\}]$
$\mathsf{E}_2 = [\{\min\}]$

$\mathsf{I}_2 = [\{\mathrm{id}\}]$

Complexity of CSP(R)

Boolean domain ($D = \{0, 1\}$)

- Infinitely many clones, but only countably many
- Every clone has a finite basis.
- Complete complexity classification [Schaefer 1978]:
  If R is 0-valid, 1-valid, Horn, anti-Horn, bijunctive, or affine,
  then CSP(R) is polynomial,
  otherwise it is NP-complete.

Finite domain of higher cardinality ($2 < |D| < \infty$)

- Uncountably many clones
- Bases are infinite in general.
- Complete complexity classification [Bulatov 2017, Zhuk 2017]:
  If R has a weak near-unanimity operation as polymorphism,
  then CSP(R) is polynomial,
  otherwise it is NP-complete.

Back to the original problem

Suppose we have a map with whisky bars marked on it.

How difficult is it to find the next whisky bar

- . . . if you are anywhere on the map?
- . . . if you are already in a whisky bar?

How difficult is it to determine the minimal distance between any two whisky bars?

Identify space with $\{0, 1\}^n$.

Specify the locations of bars by Boolean constraints using relations in R.

Measure distance using the Hamming distance.

The search for whisky bars in logical terms

NearestSolution(R), NSOL(R)

Instance: an R-formula φ and an assignment m
Solution: an assignment m' satisfying φ
Objective: minimize the Hamming distance $\mathrm{hd}(m, m')$

NearestOtherSolution(R), NOSOL(R)

Instance: an R-formula φ and an assignment m satisfying φ
Solution: an assignment $m' \neq m$ satisfying φ
Objective: minimize the Hamming distance $\mathrm{hd}(m, m')$

MinSolutionDistance(R), MSD(R)

Instance: an R-formula φ
Solution: two assignment $m' \neq m$ satisfying φ
Objective: minimize the Hamming distance $\mathrm{hd}(m, m')$

NSOL($\{x \vee y\}$)

Instance: $\varphi = (u \vee v) \wedge (u \vee w), \quad m = \{u \mapsto 0,\ v \mapsto 1,\ w \mapsto 0\}$

Optimal solution: $m' = \{u \mapsto 1,\ v \mapsto 1,\ w \mapsto 0\}$
$\mathrm{hd}(m, m') = 1$

NOSOL($\{x \vee y\}$)

Instance: $\varphi = (u \vee v) \wedge (u \vee w), \quad m = \{u \mapsto 1,\ v \mapsto 1,\ w \mapsto 0\}$

Optimal solution: $m' = \{u \mapsto 1,\ v \mapsto 0,\ w \mapsto 0\}$
$\mathrm{hd}(m, m') = 1$

MSD($\{x \vee y\}$)

Instance: $\varphi = (u \vee v) \wedge (u \vee w)$

Opt. sol.: $m = \{u \mapsto 1,\ v \mapsto 1,\ w \mapsto 0\}$, $m' = \{u \mapsto 1,\ v \mapsto 0,\ w \mapsto 0\}$
$\mathrm{hd}(m, m') = 1$

Relevance

Natural generalization of frequently used problems

E.g., NSOL is a generalization of MinOnes (find a solution with the smallest number of ones)

Related to coding theory

The solutions of the formula are the code words.
NSOL corresponds to error correction.
MSD corresponds to computing the error correcting abilities of a code.

Interesting reference problems for analyzing the complexity of other problems.

One of the first (the first?) application of weak bases

Differences to CSP(R)

Optimization instead of decision problems

- More complex notion of reduction
- More distinctions than just P and NP-complete

Standard clone theory not sufficient

- NSOL compatible with $\exists$, $\wedge$, and $=$ (though proof for $=$ non-trivial)
- NOSOL and MSD compatible with $\wedge$ and $=$, but a-priori not with $\exists$ $\Longrightarrow$ clones of partial functions and minimal weak bases needed

Classes of optimization problems

NPO: A feasible solution can be verified in polynomial time.

PO: An optimal solution can be computed in polynomial time.

A solution is ϱ-approximate if

$$\max\left(\frac{|\text{solution}|}{|\text{opt.solution}|}, \frac{|\text{opt.solution}|}{|\text{solution}|}\right) \leq \varrho$$

APX: A ϱ-approximate solution can be computed in polynomial time, where ϱ is a constant independent of the instance size.

poly-APX: A ϱ-approximate solution can be computed in polynomial time, where ϱ is a polynomial in the instance size.

PO $\subseteq$ APX $\subseteq$ poly-APX $\subseteq$ NPO

Approximation preserving reductions

$\Pi_1 \leq_{\mathrm{AP}} \Pi_2$: problem Π_1 is AP-reducible to problem Π_2 if there are polynomial-time computable functions f, g and a constant α such that:

- For all instances x of Π_1,
  $f(x)$ is an instance of Π_2.
- For all instances x of Π_1 and all solutions y of $f(x)$,
  $g(x, y)$ is a solution for x.
- For all instances x of Π_1 and all $r > 1$,
  if y is an ϱ-approximate solution for $f(x)$,
  then $g(x, y)$ is a $(1 + (\varrho - 1)\alpha + o(1))$-approximate one for x.

$\Pi_1 \equiv_{\mathrm{AP}} \Pi_2$: $\Pi_1 \leq_{\mathrm{AP}} \Pi_2$ and $\Pi_2 \leq_{\mathrm{AP}} \Pi_1$

Π is APX-hard if every problem in APX AP-reduces to Π.

Π is APX-complete, if it is APX-hard and belongs to APX.

If $\Pi_1 \leq_{\mathrm{AP}} \Pi_2$ and Π_2 is in APX, then Π_1 is also in APX.
If $\Pi_1 \leq_{\mathrm{AP}} \Pi_2$ and Π_1 is APX-hard, then Π_2 is APX-hard.

Weak partial relational clones

$\langle \Gamma \rangle_{\nexists}$: smallest set of relations

- containing Γ and the equality relation
- closed under conjunction ~~and existential quantification~~.

If $R \subseteq \langle R' \rangle_{\nexists}$, then $\mathrm{NOSOL}(R) \leq_{\mathrm{AP}} \mathrm{NOSOL}(R')$.

A partial function f is a partial polymorphism of a relation r, if $f(r, \ldots, r) \subseteq r$.

$\mathrm{pPol}(r)$: set of all partial polymorphisms of r

$\mathrm{pPol}(R) = \bigcap_{r \in R} \mathrm{pPol}(r)$

Galois connection between weak partial relational clones and strong partial clones [Romov 1981]

$\langle R \rangle_{\nexists} \subseteq \langle R' \rangle_{\nexists}$ if and only if $\mathrm{pPol}(R') \subseteq \mathrm{pPol}(R)$.

Weak base of relational clones

$\mathcal{I}(C) = \{ R \mid R = \langle R \rangle_{\nexists} \text{ and } \langle R \rangle = C \}$
interval of weak partial relational clones within the relational clone C

$\mathcal{I}_\cap(C) = \bigcap_{R \in \mathcal{I}(C)} R$

Γ is a weak base of a relational clone C, if $\mathrm{pPol}(\Gamma) = \mathrm{pPol}(\mathcal{I}_\cap(C))$.

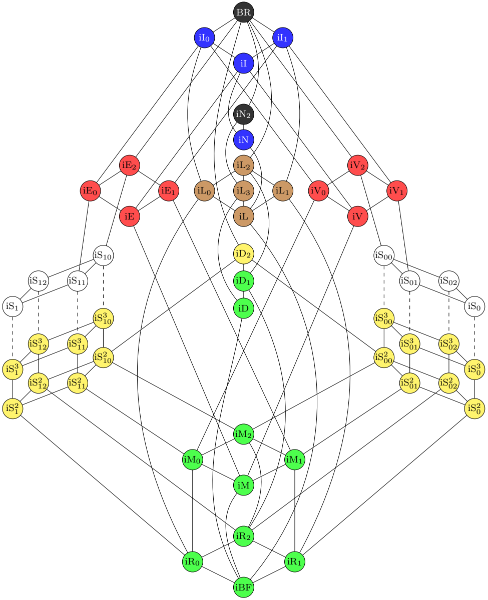
If Γ is a weak base of C, then $\Gamma \subseteq \langle \Gamma' \rangle_{\nexists}$ for any base Γ' of C.
[Schnoor, Schnoor 2008]

[Lagerkvist 2014]: Minimal weak bases for all Boolen relational clones

Relational clone iN

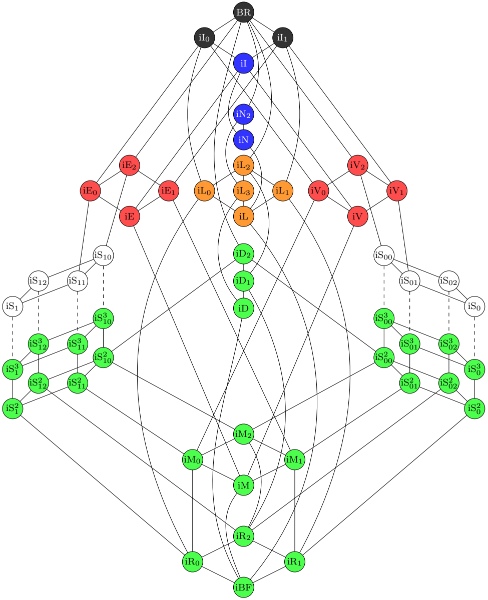
Standard base: $\{\mathrm{dup}_3\}$, where $\mathrm{dup}_3 = \{0,1\}^3 - \{010, 101\}$
Minimal weak base: $\{\mathrm{even}(x_1, x_2, x_3, x_4) \wedge (x_1 \wedge x_4 \equiv x_2 \wedge x_3)\}$

Finding the next whisky bar from anywhere (NSOL)



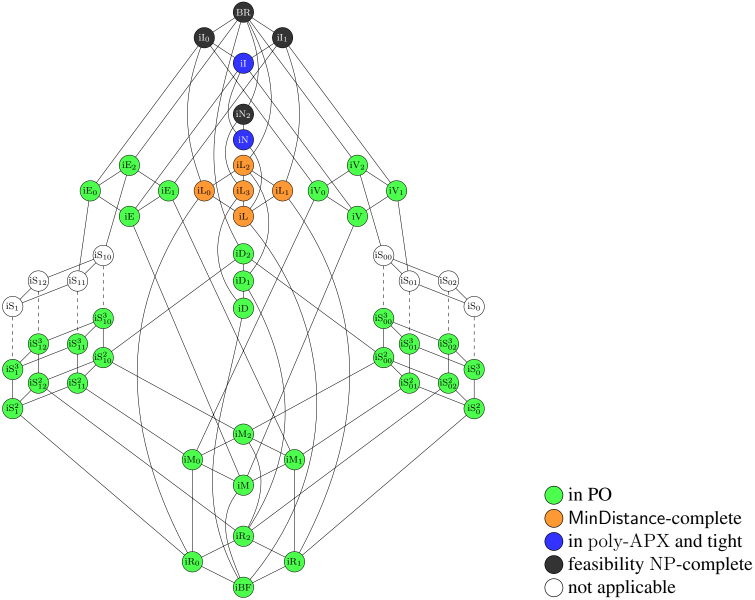
- in PO
- APX-complete
- NearestCodeword-complete
- MinHornDeletion-complete
- poly-APX-complete
- feasibility NP-complete
- not applicable

Finding the next whisky bar from a bar (NOSOL)



- in PO
- MinDistance-complete
- MinHornDeletion-complete
- in poly-APX and tight
- feasibility NP-complete
- not applicable

Determining the minimal distance between bars (MSD)



Alabama Song

Well, show me the way
To the next whisky bar
Oh, don't ask why
Oh, don't ask why

For if we don't find
The next whisky bar
I tell you we must die
I tell you we must die
I tell you, I tell you
I tell you we must die

. . .

Lyrics in German: Bertolt Brecht, 1925
English translation: Elisabeth Hauptmann
Music: Kurt Weill, 1927
Covered by Jim Morrison and The Doors, 1966